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On two integro-differential equations arising in particle transport theory

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Abstract. Two integro-differential equations arising in particle transport theory are solved explicitly using a technique involving difference equations. The physical problems to which these equations apply concern the energy–time and energy–space distributions of fast particles (neutrons, atoms, γ-rays, etc) as they slow down in a host medium. One of the equations involves the first-order derivative with respect to time or space and describes particles which scatter essentially in the forward direction. The other equation assumes a diffusive motion with almost isotropic scattering and hence involves a second-order space derivative.

Solutions are obtained in heterogeneous media where the number density of scatterers varies continuously in space and also for a series of contiguous slabs in which the material properties remain constant but change discontinuously from slab to slab.

The slowing-down density and energy deposition functions are discussed and evaluated explicitly in some special cases.

1. Introduction

The statistical distribution of free particles in a scattering and absorbing medium can be described to a high degree of accuracy by the Boltzmann transport equation (Ferziger and Kaper 1972). In its linearised form this equation is used to calculate neutron distributions in nuclear reactors (Davison 1957), displacement damage due to cascades of fast atoms (Leibfried 1965), γ-ray penetration (Fano *et al* 1959), rarefied gas flows (Cercignani 1975) and many other related problems (Allis 1956, Bharucha-Reid 1960).

In general, when the appropriate boundary conditions are given, the resulting mathematical problem is difficult to solve analytically and recourse must be made to direct numerical or other approximate methods. It is important therefore to establish for certain ideal but nevertheless realistic situations some exact analytical solutions of the Boltzmann equation. These solutions may then be used as benchmarks against which approximate methods may be calibrated and assessed. This philosophy is not confined to transport theory but it is particularly important in that field because of the unusual complexity and singular nature of the transport equation.

In this paper we shall discuss two equations which describe the slowing down of fast particles in a medium which can have a certain degree of heterogeneity, that is, the density of scattering centres may vary with space or time. The dependent variable is the density of particles $N(E, t)$ where $N(E, t)\,dE\,dt$ is the number of particles with energies between E and $E+dE$ whose t coordinate lies between t and $t+dt$. The variable t can represent space or time depending upon the physical situation under consideration. We shall be particularly concerned with the infinite medium problem of a pulsed source, that is the evolution of the energy spectrum as a function of time following the release of a mono-energetic, instantaneous burst of particles. Physically, this may correspond to neutrons or fast atoms slowing down in a host medium. However, the problem may be reinterpreted by associating t with the spatial variable (say z). Then we have the problem of mono-energetic fast particles emitted by a plane source and $N(E, t)$ describes the spatial variation of the energy spectrum. This problem is of great importance in neutron and γ-ray shielding (Fano *et al* 1959) and also in ion implantation and radiation damage (Williams 1979). However, in making the association between t and z, we calculate the so-called 'path length' distribution function. This measures the total distance travelled by particles through a zig-zag path rather than the projection of that distribution on the z axis. For many problems of practical interest, the scattering process is very highly biased in the forward direction and there is often little difference between a path length and its projection. For this reason, such a solution will have direct physical value. However, not all scattering laws possess this forward nature and in some situations it is found that a diffusive motion exists in which the average motion is almost isotropic. The spatial operator in the Boltzmann equation is then changed to a second-order differential one governed by an appropriate diffusion coefficient. This raises new mathematical problems but we shall see that the technique proposed herein, which is based upon the use of difference equations, can easily treat this new term.

2. The basic equations for study

It was stated in the Introduction that the dependent variable under study was the particle density $N(E, t)$. In fact, we find it more convenient to change to a different dependent variable and a different independent one. Thus we introduce the lethargy u defined by

$$u = \ln(E_0/E) \tag{1}$$

where E_0 is the source particle energy. We also consider the new dependent variable

$$v\Sigma(E, t)N(E, t) = \Psi(E, t) \tag{2}$$

which is the collision density since $\Sigma(E, t)$ is the scattering cross section and v is the velocity corresponding to energy E. Now changing to the variable u by

$$\Phi(u, t) = \Psi(E, t)|\mathrm{d}E/\mathrm{d}u| \tag{3}$$

we find that the basic equation for study is (Williams 1966)

$$\frac{\partial}{\partial t}\tau(u, t)\Phi(u, t) + \frac{\partial}{\partial u}(\Lambda(u, t)\Phi(u, t)) + \Phi(u, t)$$
$$= \int_0^u \mathrm{d}u'\, K(u-u'; t)\Phi(u', t) + \delta(u)\delta(t). \tag{4}$$

In this equation we have defined

$$\tau(u, t) = \frac{1}{v\Sigma(u, t)} \tag{5}$$

which is the mean time between collisions, and

$$\Lambda(u, t) = \frac{S(u, t)\,\mathrm{e}^{u}}{E_0\Sigma(u, t)} \tag{6}$$

where $S(u, t)$ is the electronic stopping cross section, i.e. a frictional term arising from the small but finite energy losses caused by electron excitation (Leibfried 1965).

Finally, we have $K(u-u'; t)$ which is the probability of a lethargy change $(u-u')$ when a particle collides with a scattering centre. The dependence of $K(u-u'; t)$ on $(u-u')$ alone is in itself an assumption but one which is frequently acceptable.

We have written equation (4) as it applies to the time-dependent problem but, as we stated earlier, it may be reinterpreted if we make the replacement

$$\frac{\partial}{\partial t}\tau(u, t)\Phi(u, t) \Rightarrow \frac{\partial}{\partial z}\frac{\Phi(u, z)}{\Sigma(u, z)} \tag{7}$$

and change t to z throughout. Then we have the distribution from a plane mono-energetic source.

If the diffusion approximation is employed then we make the replacement (Fano *et al* 1959)

$$\frac{\partial}{\partial z}\frac{\Phi(u, z)}{\Sigma(u, z)} \Rightarrow -\frac{1}{3}\frac{\partial}{\partial z}\frac{1}{\Sigma(u, z)}\frac{\partial}{\partial z}\frac{\Phi(u, z)}{\Sigma(u, z)}. \tag{8}$$

Equation (4) with the two modifications implied by (7) and (8) are the basic equations for study. Equations (4) and (7) we will designate as type A and equation (8) will be type B.

3. General solutions

In order to obtain closed form mathematical solutions it is necessary to introduce the following functional dependence on lethargy of the parameters in equation (4), viz:

$$\Sigma(u, t) = \Sigma_0(t)\,\mathrm{e}^{-ku} \tag{9}$$

and

$$S(u, t) = S_0(t)\,\mathrm{e}^{-\mu u}. \tag{10}$$

We also write

$$v(u) = v_0\,\mathrm{e}^{-\lambda u} \tag{11}$$

for the velocity. Clearly $\lambda = \frac{1}{2}$ for equation (4) but if we set $\lambda = 0$ then the modification in equation (7) can be readily included. We see therefore that

$$\tau(u, t) = \tau_0(t)\,\mathrm{e}^{(\lambda-k)u} \tag{12}$$

and

$$\Lambda(u, t) = \Lambda_0(t)\,\mathrm{e}^{-(\mu+k-1)u}. \tag{13}$$

3.1. Solution of type A equations

Inserting equations (12) and (13) into equation (4) and defining the Laplace transform

$$\bar{\Phi}(s,t)=\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-su}\Phi(u,t) \tag{14}$$

we see that the following differential-difference, or functional, equation arises:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_0(t)\bar{\Phi}(s+k-\lambda,t)+s\Lambda_0(t)\bar{\Phi}(s+\mu+k-1,t)+[1-\bar{K}(s;t)]\bar{\Phi}(s,t)=\delta(t). \tag{15}$$

We consider two possible cases: case (i) $\lambda=1-\mu$ and case (ii) $\mu+k=1$. The reader should note that these cases are not always physically realisable.

Case (i) allows us to write equation (15) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_0(t)\bar{\Phi}(s+k-\lambda,t)+\bar{H}(s;t)\bar{\Phi}(s,t)+s\Lambda_0(t)\bar{\Phi}(s+k-\lambda,t)=\delta(t) \tag{16}$$

where

$$\bar{H}(s;t)=1-\bar{K}(s;t)$$

and case (ii) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_0(t)\bar{\Phi}(s+k-\lambda,t)+[\bar{H}(s;t)+s\Lambda_0(t)]\bar{\Phi}(s,t)=\delta(t). \tag{17}$$

Case (i).

We consider case (i) and write

$$\bar{\Phi}(s,t)=P(s,t)\chi(s,t) \tag{18}$$

where $P(s,t)$ is a function to be defined. Inserting equation (18) into (16) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_0(t)\chi(s+k-\lambda,t)+\tau_0(t)\chi(s+k-\lambda,t)\frac{\mathrm{d}}{\mathrm{d}t}\log P(s+k-\lambda,t)$$
$$+\chi(s,t)+s\Lambda_0(t)\chi(s+k-\lambda,t)=\frac{\delta(t)}{P(s+k-\lambda,t)} \tag{19}$$

where $P(s,t)$ has been defined as the solution of the difference equation (Waller 1966)

$$P(s+k-\lambda,t)=\bar{H}(s;t)P(s,t). \tag{20}$$

We now assume that $H(s;t)$ does not depend on t. This is a physically reasonable assumption if it is assumed that the heterogeneity in the medium has arisen from density changes rather than changes in the scattering properties of individual scattering centres. With this assumption, $P(s,t)=P(s)$ and we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_0(t)\chi(s+k-\lambda,t)+\chi(s,t)+s\Lambda_0(t)\chi(s+k-\lambda,t)=\frac{\delta(t)}{P(s+k-\lambda)}. \tag{21}$$

Now, introduce the transforms

$$\chi(s,t)=\int_0^\infty \mathrm{d}w\, \mathrm{e}^{-sw}\chi_{\mathrm{F}}(w,t) \tag{22}$$

and

$$\chi_F(w,t)=\frac{1}{2\pi i}\int_L ds\, e^{sw}\chi(s,t). \tag{23}$$

Applying (23) to equation (21) we find after some algebra

$$\frac{\partial}{\partial t}\tau_0(t)\chi_F(w,t)+[e^{(k-\lambda)w}+(\lambda-k)\Lambda_0(t)]\chi_F(w,t)$$

$$+\Lambda_0(t)\frac{\partial}{\partial w}\chi_F(w,t)=\delta(t)\frac{1}{2\pi i}\int_L \frac{e^{pw}\,dp}{P(p)}. \tag{24}$$

Equation (24) is a partial differential equation of first order and may be solved easily by the method of characteristics. We find

$$\chi_F(w,t)=\frac{1}{\tau_0(t)}\exp[-(\lambda-k)m(t,0)]\exp\{-e^{(k-\lambda)w}\int_0^t\frac{dt'}{\tau_0(t')}\exp[-(k-\lambda)m(t,t')]\}$$

$$\times\frac{1}{2\pi i}\int_L\frac{dp}{P(p)}e^{p(w-m(t,0))} \tag{25}$$

where

$$m(t,t')=\int_{t'}^{t}\frac{\Lambda_0(t'')}{\tau_0(t'')}\,dt''. \tag{26}$$

Inverting the transform we obtain

$$\Phi(u,t)=\frac{e^{-(\lambda-k)m(t,0)}}{\tau_0(t)}\frac{1}{2\pi i}\int_L e^{s(u-m(t,0))}P(s)$$

$$\times\int_0^\infty dw_0\,e^{-sw_0}\exp(-e^{(k-\lambda)w_0}G(t))\frac{1}{2\pi i}\int_{L'}\frac{dp\,e^{pw_0}}{P(p)} \tag{27}$$

where

$$G(t)=\int_0^t\frac{dt'}{\tau_0(t')}\exp[(k-\lambda)m(t',0)]. \tag{28}$$

Now for $k>\lambda$, we may rearrange a term in equation (27) such that

$$\exp(-e^{(k-\lambda)w_0}G(t))\equiv e^{-G(t)}\,e^{-(k-\lambda)w_0}\frac{\exp[-G(t)Z/(1-Z)]}{1-Z} \tag{29}$$

where

$$Z=1-e^{-(k-\lambda)w_0}. \tag{30}$$

But the term involving Z in equation (29) is the generating function for Laguerre polynomials. Thus we may write

$$\exp(-e^{(k-\lambda)}G(t))$$

$$=e^{-G(t)}\,e^{-(k-\lambda)w_0}\sum_{n=0}^{\infty}L_n(G(t))\sum_{\nu=0}^{n}\binom{n}{\nu}(-1)^\nu e^{-\nu(k-\lambda)w_0}. \tag{31}$$

Inserting (31) into (27) and carrying out the integration over w_0 leads to

$$\Phi(u,t)=\frac{\exp[-(\lambda-k)m(t,0)-G(t)]}{\tau_0(t)}\sum_{n=0}^{\infty}\mathrm{L}_n(G(t))$$
$$\times\sum_{\nu=0}^{n}\binom{n}{\nu}(-1)^{\nu}\frac{1}{2\pi\mathrm{i}}\int_L \mathrm{d}s\,\mathrm{e}^{s(u-m(t,0))}\frac{P(s)}{P(s+(\nu+1)(k-\lambda))}. \tag{32}$$

It is not necessary to obtain $P(s)$ explicitly here, although in some simple cases it may be preferable to do so. We can instead note from the difference equation (20) that

$$\frac{P(s)}{P(s+(\nu+1)(k-\lambda))}=\prod_{l=0}^{\nu}\frac{1}{\bar{H}(s+l(k-\lambda))}. \tag{33}$$

We then have a complete solution in the sense of quadratures. It is useful to note that when $\Lambda_0=0$ and the parameter τ_0 is independent of t the solution reduces to

$$\Phi(u,t)=\frac{\mathrm{e}^{-t/\tau_0}}{\tau_0}\sum_{n=0}^{\infty}\mathrm{L}_n(t/\tau_0)\sum_{\nu=0}^{n}\binom{n}{\nu}(-1)^{\nu}\frac{1}{2\pi\mathrm{i}}\int_L\mathrm{d}s\,\mathrm{e}^{su}\prod_{l=0}^{\nu}\frac{1}{\bar{H}(s+l(k-\lambda))}. \tag{34}$$

When $k=\lambda$ a particularly simple form arises, viz:

$$\Phi(u,t)=\frac{1}{2\pi\mathrm{i}}\int_L\mathrm{d}s\,\exp\left(su-\frac{t}{\tau_0}\bar{H}(s)\right) \tag{35}$$

which could have been obtained directly from equation (15).

If $k<\lambda$ a different approach must be used. Then we must write

$$\exp(-\mathrm{e}^{(k-\lambda)w_0}G(t))\equiv\mathrm{e}^{-\tilde{G}(t,u)}\,\mathrm{e}^{-(\lambda-k)(u-w_0)}\frac{\exp[-\tilde{G}(t,u)\tilde{Z}/(1-\tilde{Z})]}{1-\tilde{Z}} \tag{36}$$

where

$$\tilde{Z}=1-\mathrm{e}^{-(\lambda-k)(u-w_0)}.$$

We then follow the same procedure as before to arrive at

$$\Phi(u,t)=\frac{\exp[-(\lambda-k)m(t,0)-\tilde{G}(t,u)]}{\tau_0(t)}\sum_{n=0}^{\infty}\mathrm{L}_n(\tilde{G}(t,u))$$
$$\times\sum_{\nu=0}^{n}\binom{n}{\nu}(-1)^{\nu}\,\mathrm{e}^{-(\nu+1)(\lambda-k)m(t,0)}\frac{1}{2\pi\mathrm{i}}$$
$$\times\int_L\mathrm{d}p\,\mathrm{e}^{p(u-m(t,0))}\frac{P(p+(\nu+1)(\lambda-k))}{P(p)}. \tag{37}$$

But the difference equation gives

$$\frac{P(p+(\nu+1)(\lambda-k))}{P(p)}=\prod_{m=1}^{\nu+1}\frac{1}{\bar{H}(p+m(\lambda-k))} \tag{38}$$

which allows us after some further manipulation to write

$$\Phi(u, t) = \frac{\exp[-(\lambda - k)u - \tilde{G}(t, u)]}{\tau_0(t)} \sum_{n=0}^{\infty} \mathrm{L}_n(\tilde{G}(t, u))$$
$$\times \sum_{\nu=0}^{n} \binom{n}{\nu} (-1)^{\nu} \, \mathrm{e}^{-(\nu+1)(\lambda-k)m(t,0)} \frac{1}{2\pi\mathrm{i}}$$
$$\times \int_L \mathrm{d}p \, \mathrm{e}^{p(u-m(t,0))} \prod_{l=0}^{\nu} \frac{1}{\bar{H}(p + l(\lambda - k))}. \tag{39}$$

Special case

As an example of this technique let us assume that $K(u) = \mathrm{e}^{-u}$, $k = 1$, $\Lambda_0 = 0$, $\lambda = 0$. Then from equation (34) we may write

$$\prod_{l=0}^{\nu} \frac{s + l + 1}{s + l} = \frac{s + \nu + 1}{s} \tag{40}$$

and

$$\frac{1}{2\pi\mathrm{i}} \int_L \mathrm{e}^{su} \left(1 + \frac{\nu + 1}{s}\right) \mathrm{d}s = \delta(u) + \nu + 1. \tag{41}$$

Hence, changing t/τ_0 to z/λ_0, we get

$$\Phi(u, z) = \frac{\mathrm{e}^{-z/\lambda_0}}{\lambda_0} \sum_{n=0}^{\infty} \mathrm{L}_n(z/\lambda_0) \sum_{\nu=0}^{n} \binom{n}{\nu} (-1)^{\nu} (\delta(u) + \nu + 1) \tag{42}$$

$$= \frac{\mathrm{e}^{-z/\lambda_0}}{\lambda_0} (\delta(u) + z/\lambda_0). \tag{43}$$

Fortuitously, this expression possesses the same energy spectrum as the infinite medium spatially independent problem. In general, however, this is not the case. Thus for a more general kernel, say

$$K(u) = \frac{1}{\gamma} \mathrm{e}^{-u/\gamma} \tag{44}$$

where $\bar{H}(s) = \gamma s (1 + \gamma s)^{-1}$ we find that (see § 5)

$$\frac{1}{2\pi\mathrm{i}} \int_L \mathrm{d}s \, \mathrm{e}^{su} \prod_{l=0}^{\nu} \frac{1}{\bar{H}(s + l(k - \lambda))}$$
$$= \delta(u) + \frac{1 + \nu}{\gamma(k - \lambda)} {}_2F_1\left(1 - \frac{1}{\gamma(k - \lambda)}, -\nu; 2; 1 - \mathrm{e}^{-u(k-\lambda)}\right) \tag{45}$$

where ${}_2F_1(\ldots)$ is the hypergeometric function.

Case (ii)

Here we return to equation (17) and note that the solution of this problem is very simple provided $\Lambda_0(t)$ and $\bar{H}(s; t)$ are independent of t. Further if we define

$$\bar{R}(s) = \bar{H}(s) + \Lambda_0 s \tag{46}$$

we see that the resulting equation has the same form as equation (16) but with $\bar{R}$ replacing $\bar{H}$ and Λ_0 set equal to zero. Equation (34) therefore applies but with

$$\frac{t}{\tau_0} \Rightarrow \int_0^t \frac{\mathrm{d}t'}{\tau_0(t')} \tag{47}$$

and

$$\bar{H}(s+l(k-\lambda)) \Rightarrow \bar{R}(s+l(k-\lambda)). \tag{48}$$

3.2. Solution of type B equation

Introducing equation (8) into (4) we find the diffusion equation

$$-\frac{1}{3\Sigma^2(u)} \frac{\partial^2}{\partial z^2}\Phi(u, z)+\Phi(u, z)+\frac{\partial}{\partial u}(\Lambda(u)\Phi(u, z)) = \int_0^u \mathrm{d}u'\, K(u-u')\Phi(u', z)+\delta(u)\delta(z). \tag{49}$$

We have assumed a homogeneous medium because it does not appear to be possible to obtain an analytical solution to equation (49) when Σ and Λ depend upon z. However, in § 3.3 below, we indicate why this is not necessarily a serious restriction.

In solving equation (49) we shall take the variation of cross sections given by (9) and (10) and in addition define L_0^2 by

$$\frac{1}{3\Sigma^2(u)} = L_0^2\, \mathrm{e}^{-2ku}. \tag{50}$$

Application of the Laplace transform in lethargy leads then to

$$-L_0^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\bar{\Phi}(s+2k, z)+s\Lambda_0\bar{\Phi}(s+\mu+k-1, z)+\bar{H}(s)\bar{\Phi}(s, z)=\delta(z). \tag{51}$$

This may be solved if $\mu+k=1$ or if $\mu-k=1$ for $\Lambda_0 \neq 0$. For $\Lambda_0=0$ there is no restriction on k. We will assume that $\mu+k=1$ and write equation (51) as

$$-L_0^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\bar{\Phi}(s+2k, z)+\bar{R}(s)\bar{\Phi}(s, z)=\delta(z). \tag{52}$$

Now introducing

$$\bar{\Phi}(s, z)=Q(s)\chi(s, z) \tag{53}$$

where $Q(s)$ is defined by

$$Q(s+2k)=\bar{R}(s)Q(s) \tag{54}$$

we find that

$$L_0^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\chi(s+2k, z)-\chi(s, z)+\frac{\delta(z)}{Q(s+2k)}=0. \tag{55}$$

Using the χ_{F} transform of equations (22) and (23) we find

$$L_0^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\chi_{\mathrm{F}}(w, z)-\mathrm{e}^{2kw}\chi_{\mathrm{F}}(w, z)+\frac{\delta(z)}{2\pi\mathrm{i}}\int_L \frac{\mathrm{e}^{qw}\, \mathrm{d}q}{Q(q)}=0 \tag{56}$$

whence for an infinite medium

$$\Phi(u, z) = \frac{1}{2L_0} \frac{1}{2\pi i} \int_L ds\, e^{su} Q(s)$$
$$\times \int_0^\infty dw\, e^{-(s+k)w} \exp\left(-\frac{|z|}{L_0} e^{kw}\right) \frac{1}{2\pi i} \int_{L'} \frac{e^{qw}}{Q(q)} dq. \tag{57}$$

In order to cast equation (57) into a suitable form for calculation we take the Fourier transform with respect to z, whence it becomes

$$\bar{\Phi}(u, \xi) = \frac{1}{2\pi i} \int_L ds\, e^{su} Q(s) \int_0^\infty \frac{dw\, e^{-w(s+k)}}{1 + \xi^2 L_0^2 e^{-2kw}} \frac{1}{2\pi i} \int_{L'} \frac{e^{qw}\, dq}{Q(q)}. \tag{58}$$

Now we rearrange the term involving ξ^2 such that

$$\frac{1}{1 + \xi^2 L_0^2 e^{-2kw}} = \frac{1}{1 + \xi^2 L_0^2 - (1 - e^{-2kw}) \xi^2 L_0^2}$$
$$= \frac{1}{1 + \xi^2 L_0^2} \sum_{n=0}^\infty \frac{\xi^{2n} L_0^{2n}}{(1 + \xi^2 L_0^2)^n} \sum_{\nu=0}^n \binom{n}{\nu} (-1)^\nu e^{-2\nu k w}. \tag{59}$$

But clearly

$$\frac{1}{2\pi} \int_{-\infty}^\infty \frac{d\xi\, e^{i\xi z} \xi^{2n} L_0^{2n}}{(1 + \xi^2 L_0^2)^{n+1}} = \frac{e^{-|z|/L_0}}{2L_0} \mathscr{L}_n\left(\frac{|z|}{L_0}\right) \tag{60}$$

where $\mathscr{L}_n(y)$ are a set of polynomials. The $\mathscr{L}_n(y)$ are unrelated to any standard polynomials but the first few are

$$\begin{aligned} \mathscr{L}_0(y) &= 1 \\ \mathscr{L}_1(y) &= \tfrac{1}{2}(1 - y) \\ \mathscr{L}_2(y) &= \tfrac{1}{8}(3 - 5y + y^2) \\ \mathscr{L}_3(y) &= \tfrac{1}{48}(15 - 33y + 12y^2 - y^3). \end{aligned} \tag{61}$$

Inserting (59) into (58) and using (60) we find after integrating over w that

$$\Phi(u, z) = \frac{1}{2L_0} e^{-|z|/L_0} \sum_{n=0}^\infty \mathscr{L}_n\left(\frac{|z|}{L_0}\right) \sum_{\nu=0}^n \binom{n}{\nu} (-1)^\nu \frac{1}{2\pi i} \int_L ds\, e^{su} \prod_{l=0}^\nu \frac{1}{\bar{R}(s + 2lk)}. \tag{62}$$

When $\Lambda_0 = 0$ and $K(u) = e^{-u/\gamma}/\gamma$ we may write the inversion integral as in equation (45) but with $(k - \lambda)$ replaced by $2k$. It is therefore obvious that in the special case of $\gamma = 1$ and $k = \frac{1}{2}$ the solution assumes the form

$$\Phi(u, z) = \frac{1}{2L_0} e^{-|z|/L_0} \left[\delta(u) + \tfrac{1}{2}\left(1 + \frac{|z|}{L_0}\right)\right]. \tag{63}$$

3.3. A repeating slab structure

Although the degree and nature of the heterogeneity in the t or z variables are limited in the above solutions, it is possible to extend the usefulness of the homogeneous medium solutions. This can be done in the particular case of a repeating slab structure where the properties of the medium change in a stepwise fashion from slab to slab. In fact, this situation is very common in nuclear reactors and in ion implantation problems (Williams 1971, 1979) and so is worthy of discussion.

We consider the homogeneous form of equation (4) with $t/\tau(u)$ replaced by $z\Sigma(u)$ and $\lambda=0$. In this case the source will be located in one slab only and the solution will be given by equation (27) with

$$m(z,0)=\Lambda_0\Sigma_0 z \tag{64}$$

and

$$G(z)=\frac{1}{k\Lambda_0}(e^{k\Lambda_0\Sigma_0 z}-1). \tag{65}$$

This solution, which we call $\Phi_1(u,z)$, will be valid until the new material is reached say at $z=0$. Then the new equation to be solved is the homogeneous form of equation (4) with the new material properties and the boundary condition of continuity of particle density, i.e.

$$N_1(u,a)=N_2(u,a) \tag{66}$$

where the subscripts 1 and 2 refer to the media on each side of the discontinuity. To obtain $N_2(u,z)$ we solve equation (4) in the same manner as before up to equation (24). In the region $z>a$, equation (34) is solved with no source term and we find

$$\chi_2(s,z)=\int_0^\infty dw\, e^{-sw}\chi_{F_2}(w,a)\exp\left(\Lambda_{02}\Sigma_{02}k_2 z-\frac{e^{k_2 w}}{k_2\Lambda_{02}}(1-e^{-k_2\Lambda_{02}\Sigma_{02}z})\right); \tag{67}$$

but from the boundary condition (66) we may write

$$\Sigma_{02}\bar{\Phi}_1(s+k_1-k_2,a)=\Sigma_{01}\bar{\Phi}_2(s,a). \tag{68}$$

Thus

$$\chi_{F_2}(w,a)=\frac{1}{2\pi i}\int_L ds\, e^{sw}\frac{\Sigma_{02}}{\Sigma_{01}}\frac{\bar{\Phi}_1(s+k_1-k_2,a)}{P_2(s)} \tag{69}$$

and the distribution in $z>a$ can be written

$$\Phi_2(u,z)=\frac{1}{2\pi i}\int_L ds\, e^{su}P_2(s)\int_0^\infty dw\, e^{-sw}\exp\left(\Lambda_{02}\Sigma_{02}k_2 z-\frac{e^{k_2 w}}{k_2\Lambda_{02}}(1-e^{-k_2\Lambda_{02}\Sigma_{02}z})\right)$$
$$\times\frac{1}{2\pi i}\int_{L'} ds'\, e^{s'w}\frac{\Sigma_{02}}{\Sigma_{01}}\frac{\bar{\Phi}_1(s'+k_1-k_2,a)}{P_2(s')}. \tag{70}$$

Since we know $\Phi_1(s,a)$ we have the complete solution. This procedure may be carried through for any number of contiguous slabs. A practical example of this technique has been discussed by the author (Williams 1978).

4. The slowing-down density and energy deposition

In the physics of particle slowing down, two important quantities arise. They are the slowing-down density $q(u,t)$ which is the number of particles crossing the lethargy u per unit time, and the slowing-down energy density $W(u,t)$, which is the amount of energy possessed by the particles crossing lethargy u per unit time.

By definition we may write (Williams 1966):

$$q(u, t)=\int_{-\infty}^{u} \mathrm{d}u' \,\Phi(u', t) \int_{u}^{\infty} \mathrm{d}u'' \,K(u''-u') \tag{71}$$

and

$$W(u, t)=E_0 \int_{-\infty}^{u} \mathrm{d}u' \,\Phi(u', t) \int_{u}^{\infty} \mathrm{d}u'' \,\mathrm{e}^{-u''} K(u''-u'). \tag{72}$$

In terms of Laplace transforms these quantities assume the form

$$s\bar{q}(s, t)=\bar{H}(s)\bar{\Phi}(s, t) \tag{73}$$

and

$$s\bar{W}(s, t)=\bar{H}(s+1)\bar{\Phi}(s+1, t). \tag{74}$$

For some practical problems the values of $q(u, t)$ and $W(u, t)$ are required at zero energy (or infinite lethargy). In fact, $q(\infty, t)$ corresponds, in the case of slowing-down atoms in a host medium, to the profile of stopped or implanted atoms. Similarly $W(\infty, t)$ corresponds to the total energy deposited to the host medium by the slowing-down particles. We see therefore from the properties of Laplace transforms that

$$q(\infty, t)=\lim_{s\to 0} s\bar{q}(s, t) \tag{75}$$

and

$$W(\infty, t)=\lim_{s\to 0} s\bar{W}(s, t). \tag{76}$$

As an example of this technique consider equation (27), where

$$s\bar{q}(s, t)=\frac{\mathrm{e}^{-(\lambda-k)m(t,0)}}{\tau_0(t)}\bar{H}(s)P(s) \times \int_0^{\infty} \mathrm{d}w_0 \,\mathrm{e}^{-sw_0} \exp(-\mathrm{e}^{(k-\lambda)w_0} G(t)) \frac{1}{2\pi\mathrm{i}} \int_L \frac{\mathrm{d}p \,\mathrm{e}^{pw_0}}{P(p)}. \tag{77}$$

Since $\bar{H}(s)P(s)=P(s+k-\lambda)$, we can write immediately

$$q(\infty, t)=\frac{\mathrm{e}^{-(\lambda-k)m(t,0)}}{\tau_0(t)}P(k-\lambda) \int_0^{\infty} \mathrm{d}w_0 \exp[-\mathrm{e}^{(k-\lambda)w_0} G(t)] \frac{1}{2\pi\mathrm{i}} \int_L \frac{\mathrm{d}p \,\mathrm{e}^{pw_0}}{P(p)}. \tag{78}$$

In fact $q(\infty, t)$ is only bounded for $k>\lambda$ for obvious physical reasons.

Similarly we may write

$$W(\infty, t)=\frac{E_0 \,\mathrm{e}^{-(\lambda-k)m(t,0)}}{\tau_0(t)}P(k+\lambda-1) \times \int_0^{\infty} \mathrm{d}w_0 \,\mathrm{e}^{-w_0} \exp(-\mathrm{e}^{(k-\lambda)w_0} G(t)) \frac{1}{2\pi\mathrm{i}} \int_L \frac{\mathrm{d}p \,\mathrm{e}^{pw_0}}{P(p)} \tag{79}$$

where $k>\lambda-1$.

These expressions for $q(\infty, t)$ and $W(\infty, t)$ may be written conveniently in terms of the Laguerre polynomial expansion discussed earlier. The nice aspect of these results is that no Laplace inversion is necessary.

5. The difference equation

It is clear that the role played by the solution of the difference equation is a crucial one and therefore some investigation of this function will be worthwhile. The equation for study is therefore

$$P(s+h)=\bar{H}(s)P(s). \tag{80}$$

The theory of first-order difference equations of this type is fully discussed by Milne-Thomson (1933) and Levy and Lessman (1959) and the results presented here are based upon the work of these authors.

We note first that for values of s which are integral values of h, i.e. nh where n is an integer, we may obtain by induction the following results:

$$\frac{P(s)}{P(s+(n+1)h)}=\prod_{l=0}^{n}\frac{1}{\bar{H}(s+lh)} \tag{81}$$

$$\frac{P(s-(n+1)h)}{P(s)}=\prod_{l=1}^{n+1}\frac{1}{\bar{H}(s-lh)}. \tag{82}$$

These results have been used in equations (33) and (38) above.

If a general expression for $P(s)$ is desired the problem is generally more difficult. However, there are some special cases worthy of discussion. For example, if $\bar{H}(s)$ is given by

$$\bar{H}(s)=a\frac{\Pi_{l=1}^{q}(s-\alpha_l)}{\Pi_{m=1}^{p}(s-\beta_m)}, \tag{83}$$

then a general solution of (80), apart from an arbitrary factor, is

$$P(s)=(ah^{q-p})^{s/h}\frac{\Pi_{l=1}^{q}\Gamma[(s-\alpha_l)/h]}{\Pi_{m=1}^{p}\Gamma[(s-\beta_m)/h]}. \tag{84}$$

We see therefore that in the special case considered earlier where

$$\bar{H}(s)=\frac{\gamma s}{1+\gamma s} \tag{85}$$

and $h=k-\lambda$

$$P(s)=\frac{\Gamma[s/(k-\lambda)]}{\Gamma\{s/(k-\lambda)+1/[\gamma(k-\lambda)]\}}. \tag{86}$$

It was by using this expression in equation (32) that we obtained equation (45).

It is possible to obtain expressions for $P(s)$ for more general forms of $\bar{H}(s)$ provided the behaviour for large $|s|$ is known. Thus we may write

$$P(s)=\prod_{j=0}^{\infty}\frac{f(j)}{\bar{H}(s+jh)} \tag{87}$$

where $f(j)$ is a factor which ensures that

$$\lim_{j\to\infty}\frac{f(j)}{\bar{H}(jk)}\to 1. \tag{88}$$

Such a convergence factor is not too difficult to obtain. For example, in the case just considered

$$\bar{H}(s+jh) \sim 1 - 1/\gamma jh \qquad \text{as } |j| \to \infty$$
$$\simeq e^{-1/\gamma jh}.$$

Hence we may write

$$P(s) = \frac{1}{\bar{H}(s)} \prod_{j=1}^{\infty} \frac{e^{-1/\gamma jh}}{\bar{H}(s+jh)} \tag{89}$$

$$= \frac{1+\gamma s}{\gamma s} \prod_{j=1}^{\infty} \frac{(1+\gamma s+\gamma jh)}{\gamma(s+jh)} e^{-1/\gamma jh} \tag{90}$$

which may be rearranged to the form

$$P(s) = \frac{1+\gamma s}{\gamma s} \frac{\Pi_{j=1}^{\infty} e^{s/jh}/(1+s/jh)}{\Pi_{j=1}^{\infty} e^{(\gamma s+1)/\gamma jh}/[1+(1+\gamma s)/\gamma jh]}. \tag{91}$$

But by definition (Levy and Lessman 1959)

$$\Gamma(x+1) = e^{-cx} \prod_{j=1}^{\infty} \frac{e^{x/j}}{1+x/j}. \tag{92}$$

Hence equation (91) reduces to equation (86) apart from the arbitrary constant $\exp(-c/\gamma h)$. Thus by judicious choice of $f(j)$ we may obtain explicit expressions for $P(s)$; the subsequent Laplace inversion, however, is not always simple.

6. Summary and discussion

Two integro-differential equations have been solved explicitly. These equations arise in the theory of particle slowing down involving neutrons and fast atoms. Similar equations may also be found in other areas of statistical physics such as cosmic ray production, nuclear particle detectors and queueing theory (Bharucha-Reid 1960). It seems likely therefore that the techniques developed here will have wider application.

The basis of the solution method relies upon the extension of an idea due to Waller (1966). This consists of the introduction of a function defined by a difference equation which then renders the Laplace transformed equation amenable to solution by elementary methods.

It is shown how some useful physical quantities arising in transport theory, e.g. slowing-down density and energy deposition, can be calculated rather easily from the Laplace transform of the solution of the integro-differential equations.

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